

Additivity and multiplicativity properties of some Gaussian channels for Gaussian inputs

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We prove multiplicativity of maximal output p norm of classical noise channels and thermal noise channels of arbitrary modes for all $p > 1$ under the assumption that the input signal states are Gaussian states. As a direct consequence, we also show the additivity of the minimal output entropy and that of the energy-constrained Holevo capacity for those Gaussian channels under Gaussian inputs. To the best of our knowledge, newly discovered majorization relation on symplectic eigenvalues, which is also of independent interest, plays a central role in the proof.

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I. INTRODUCTION

One of the goals of quantum information theory is to clarify the ultimate capability of information processing harnessed by using quantum mechanics [1, 2]. The celebrated Holevo-Schumacher-Westmoreland theorem [3, 4] gives us a formal basis to determine the ultimate transmission rate of classical information encoded in quantum states transmitted through a quantum channel. Yet, an important question is still unanswered in terms of the classical capacity of quantum channels. It is the additivity question; Do the entangled inputs over several invocations of quantum channels improve the classical capacity of quantum channels? Despite many efforts devoted to the additivity problems of quantum channels, the additivity properties have been proven for a few examples, such as entanglement breaking channels [5], unital qubit channels [6], depolarizing channels [7], and contravariant channels [8]. Surprisingly, the additivity problems of quantum channels have been shown to be equivalent to the seemingly unrelated additivity problems of quantum entanglement, i.e., the additivity and the strong superadditivity of entanglement of formation [9, 10, 11, 12]. All of them are not completely solved and are now major concerns in quantum information and the quantum entanglement theories.

As for continuous-variable quantum systems, in spite of intensive research [13, 14, 15], only lossy channels have been proven to be additive [16]. The additivity problems may be much more intractable for continuous-variable quantum systems. The natural question is, therefore, what can we say about the additivity properties of Gaussian quantum channels if we restrict the input signal states to be Gaussian states? This question has its own significance. One rationale is that the Gaussian channels correspond to the so-called Gaussian operations that can be implemented by current experimental techniques, such as beamsplitters, phase shifters, squeezers, and homodyne measurements. Another is the mathematical simplicity; Gaussian operations on Gaussian states are completely characterized by finite dimensional matrices and vectors, although the underlying Hilbert space is infinite dimensional.

Due to their mathematical simplicity, the additivity problems of Gaussian channels under Gaussian inputs provide a potential firm step towards answering the additivity questions.

Serafini *et al.* [17] formulated the multiplicativity problems of the purity at the output of Gaussian channels measured by the Schatten p norm under the assumption that the input signal states were Gaussian states. In this paper, we extend their formalism to the additivity problems of minimal output entropy and energy-constrained Holevo capacity and prove the additivity properties of two classes of Gaussian channels — the classical noise channels and thermal noise channels of arbitrary modes.

The paper is organized as follows. In Sec. II we introduce the notation and present basic facts about Gaussian states and the symplectic transformations used in this paper. In Sec. III we define Gaussian channels and introduce three figures of merits to quantify Gaussian channels — the maximal output p norm, the minimal output entropy, and the Holevo capacity. In Sec. IV we formulate the additivity and multiplicativity problems of Gaussian channels for Gaussian inputs. In Sec. V we prove a trace formula for symplectic eigenvalues and a majorization relation on symplectic eigenvalues that is an immediate consequence of the trace formula. By virtue of this majorization relation on symplectic eigenvalues, we prove the additivity and multiplicativity properties of classical noise channels and thermal noise channels of arbitrary modes in Sec. VI. Section VII is devoted to concluding remarks.

II. GAUSSIAN STATES

In this section, we introduce the notation and summarize the basic facts about Gaussian states and symplectic transformations [18]. We consider an n mode quantum system, such as a radiation field. Each mode corresponds to a quantum mechanical harmonic oscillator with two canonical degrees of freedom and the quadratures of each mode correspond to the position and momentum of the harmonic oscillator. Thus an n mode state has a $2n$ canonical degrees of freedom. Let Q_k and P_k denote the “position” and “momentum” operators associated with the k th mode ($k = 1, 2, \dots, n$). These operators or canonical variables are written in terms of the creation and

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annihilation operators of the mode;

$$Q_k = \sqrt{\frac{1}{2\omega_k}} (a_k + a_k^\dagger) \quad (1)$$

and

$$P_k = -i \sqrt{\frac{\omega_k}{2}} (a_k - a_k^\dagger), \quad (2)$$

where ω_k denotes the energy of the k th mode ($\hbar = 1$). Since $[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$ and $[a_j, a_k^\dagger] = \delta_{jk}$, we have $[Q_j, Q_k] = [P_j, P_k] = 0$ and $[Q_j, P_k] = i\delta_{jk}$. Defining

$$\begin{aligned} R &= (R_1, R_2, \dots, R_{2n})^T \\ &= (\omega_1^{1/2} Q_1, \omega_1^{-1/2} P_1, \dots, \omega_n^{1/2} Q_n, \omega_n^{-1/2} P_n)^T, \end{aligned} \quad (3)$$

these canonical commutation relations (CCRs) can be written as $[R_j, R_k] = i(J_n)_{jk}$. Here, $J_n = \bigoplus_{j=1}^n J_1$ with

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4)$$

In the following, the characteristic function defined as $\chi(\xi) = \text{Tr}[\rho \mathcal{W}(\xi)]$ plays a key role. Here, $\mathcal{W}(\xi) = \exp(i\xi^T J_n R)$ is called the Weyl operators and ρ denotes the density operator. The density operator in turn can be written in terms of its characteristic function and Weyl operators as follows.

$$\rho = \frac{1}{(2\pi)^n} \int d^{2n} \xi \chi(\xi) \mathcal{W}(-\xi). \quad (5)$$

A Gaussian state is defined as a state whose characteristic function is a Gaussian function:

$$\chi(\xi) = \exp[-\frac{1}{4} \xi^T \Gamma \xi + i D^T \xi]. \quad (6)$$

Here, $\Gamma > 0$ is a real symmetric matrix and $D \in \mathbb{R}^{2n}$. The first moment is also called the displacement or mean and given by $m_j = \text{Tr}(\rho R_j)$ and the second moment is given by

$$\gamma_{jk} = 2\text{Tr}[\rho(R_j - m_j)(R_k - m_k)] - i(J_n)_{jk}, \quad (7)$$

which is called the covariance of canonical variables. The $2n \times 2n$ real symmetric matrix (γ_{jk}) is called the covariance matrix γ . Γ and D in Eq. (6) are given by $\Gamma = J_n^T \gamma J_n$ and $D = J_n m$.

Note that due to our choice of canonical variables, $R_{2j-1} = \omega_j^{1/2} Q_j$ and $R_{2j} = \omega_j^{-1/2} P_j$ ($j = 1, 2, \dots, n$), the trace of the principal submatrix of the j th mode of γ ,

$$\gamma_{[j]} = \begin{pmatrix} \gamma_{2j-1, 2j-1} & \gamma_{2j-1, 2j} \\ \gamma_{2j, 2j-1} & \gamma_{2j, 2j} \end{pmatrix} \quad (8)$$

gives the energy of the j th mode if $m_{2j-1} = m_{2j} = 0$;

$$\frac{1}{4} \omega_k (\gamma_{2j-1, 2j-1} + \gamma_{2j, 2j}) = \omega_k \left(\langle a_j^\dagger a_j \rangle + \frac{1}{2} \right). \quad (9)$$

A density operator ρ is a positive semidefinite operator ($\rho \geq 0$) with $\text{Tr} \rho = 1$. The necessary and sufficient condition for

$\rho \geq 0$ in a Gaussian state is given in terms of the covariance matrix as follows [19].

$$\gamma + iJ_n \geq 0. \quad (10)$$

Furthermore, the necessary and sufficient condition for a pure Gaussian state is given by [20]

$$\det \gamma = 1. \quad (11)$$

A linear transformation on canonical variables is written as $R \rightarrow R' = S R$. Since the new variables R' also must conserve the CCR $[R'_j, R'_k] = i(J_n)_{jk}$, $S J_n S^T = J_n$ must hold. Such a $2n \times 2n$ real matrix satisfying $S J_n S^T = J_n$ is called a symplectic transformation, $S \in \text{Sp}(2n, \mathbb{R}) = \{S | S J_n S^T = J_n\}$, which forms a group so that S^{-1} and $S_1 S_2$ are symplectic if $S, S_1, S_2 \in \text{Sp}(2n, \mathbb{R})$. Furthermore, S^T is also symplectic and $\det S = 1$ [21] if $S \in \text{Sp}(2n, \mathbb{R})$. In this paper, we repeatedly use the following Williamson theorem [22]. For a real symmetric positive definite $2n \times 2n$ matrix $A = A^T > 0$, there exists a symplectic transformation $S \in \text{Sp}(2n, \mathbb{R})$ such that

$$S A S^T = \text{diag}(v_1, v_1, v_2, v_2, \dots, v_n, v_n) \quad (12)$$

with $v_j (> 0)$ called symplectic eigenvalues of A ($j = 1, 2, \dots, n$). Equation (12) is called the Williamson standard form of A . The symplectic eigenvalues can be computed via the eigenvalues of $J_n A$, which are $\pm i v_j$ ($j = 1, 2, \dots, n$). Any symplectic transformation $S \in \text{Sp}(2n, \mathbb{R})$ can be decomposed into

$$S = T^{(1)} Z T^{(2)} \quad (13)$$

with $T^{(1)}, T^{(2)} \in \text{Sp}(2n, \mathbb{R}) \cap O(2n) = K(n)$ and

$$Z = \text{diag}(z_1, z_1^{-1}, \dots, z_n, z_n^{-1}), \quad (14)$$

where $z_j \geq 1$ ($j = 1, 2, \dots, n$) [23]. $O(2n)$ denotes the orthogonal group whose elements are $2n \times 2n$ real orthogonal matrices. Equation (13) is called the Euler decomposition of symplectic transformations. $K(n)$ is a maximal compact subgroup of $\text{Sp}(2n, \mathbb{R})$ and is isomorphic to $U(n)$, the unitary group whose elements are $n \times n$ unitary matrices [24]. The isomorphism is established via the following correspondence:

$$T_{2j-1, 2k-1} = T_{2j, 2k} = \text{Re} u_{j,k} = u_{j,k}^R \quad (15)$$

and

$$T_{2j-1, 2k} = -T_{2j, 2k-1} = \text{Im} u_{j,k} = u_{j,k}^I, \quad (16)$$

where $T \in K(n)$ and $u_{j,k}$ are (j, k) components of $n \times n$ unitary matrices U . Using Eqs. (15) and (16), the isomorphism $K(n) \simeq U(n)$ is easily verified by direct calculations.

Since the covariance matrix of an n -mode state is a $2n \times 2n$ real symmetric positive-definite matrix, it can be cast into the Williamson standard form. In terms of symplectic eigenvalues, condition (10) is rephrased as $v_j \geq 1$ ($j = 1, 2, \dots, n$) and condition (11) is written as $v_j = 1$ for all j .

A canonical linear transformation corresponds to a unitary transformation in the Hilbert space. Such a unitary transformation is defined by $U_S \mathcal{W}(\xi) U_S^\dagger = \mathcal{W}(S^{-1}\xi)$, and the density operator is transformed as $\rho \mapsto U_S \rho U_S^\dagger = \tilde{\rho}$ correspondingly. It is easy to see that $U_S^\dagger = U_{S^{-1}}$. The characteristic function of the new state $\tilde{\rho}$ is given by $\text{Tr}[U_S \rho U_S^\dagger \mathcal{W}(\xi)] = \text{Tr}[\rho \mathcal{W}(S\xi)] = \chi(S\xi)$. Accordingly, the covariance matrix and the displacement are transformed as $\gamma \mapsto S^{-1}\gamma(S^{-1})^T$ and $m \mapsto S^{-1}m$. Note that the symplectic eigenvalues are invariant under such symplectic transformations on the covariance matrix.

Coherent states, squeezed states, and thermal states are typical Gaussian states, while the number states (of the single mode) given by $|k\rangle\langle k|$ with

$$|k\rangle = \frac{1}{\sqrt{k!}} (a^\dagger)^k |0\rangle, \quad k = 1, 2, \dots, \quad (17)$$

are not. However, a vacuum state $|0\rangle\langle 0|$, which is a special case of the number states, is a Gaussian state with the covariance matrix,

$$\gamma_{\text{vac}} = \text{diag}(1, 1). \quad (18)$$

This is the minimal-energy pure state. The coherent state is the displaced vacuum state so that the covariance matrix is given by Eq. (18) but has a finite displacement. The thermal state of the single mode,

$$\rho_{\text{th}} = \frac{1}{1 + \langle n \rangle} \sum_{k=0}^{\infty} \left(\frac{\langle n \rangle}{1 + \langle n \rangle} \right)^k |k\rangle\langle k| \quad (19)$$

has the covariance matrix

$$\gamma_{\text{th}} = \text{diag}(2\langle n \rangle + 1, 2\langle n \rangle + 1) \quad (20)$$

with $\langle n \rangle$ being the mean photon number of the mode.

III. GAUSSIAN CHANNELS AND THEIR QUANTIFICATION

A Gaussian channel Φ is a completely positive trace preserving map that maps Gaussian input states ρ to Gaussian output states $\Phi(\rho)$ [25, 26]. The covariance matrix is transformed according to

$$\gamma \mapsto \phi(\gamma) = X^T \gamma X + Y, \quad (21)$$

where X and Y are $2n \times 2n$ real matrices and Y is positive and symmetric ($Y = Y^T \geq 0$). The complete positivity of the channel is expressed in terms of these matrices as [27]

$$Y + iJ_n - iX^T J_n X \geq 0. \quad (22)$$

Hereafter, we write a Gaussian channel by a capital greek letter and the corresponding transformation on the covariance matrix by the corresponding lower case greek letter.

There are several figures of merits for quantifying quantum channels. Here we take three of them; the maximal output p

norm [28], the minimal output entropy [29], and the Holevo capacity [30] for Gaussian state inputs.

The Gaussian maximal output p norm is defined as

$$\xi_p(\Phi) = \sup_{\rho \in \mathcal{G}} \|\Phi(\rho)\|_p, \quad (23)$$

where $\|\rho\|_p = (\text{Tr}|\rho|^p)^{1/p}$ is the Schatten p norm ($p \geq 1$) with $|A| = \sqrt{A^\dagger A}$. In Eq. (23), \mathcal{G} denotes the set of all Gaussian states. For a Gaussian state with covariance matrix γ ,

$$\text{Tr}\rho_\gamma^p = \prod_{j=1}^n \frac{2^p}{f_p(\nu_j)} = \frac{2^{pn}}{F_p(\nu)}, \quad (24)$$

where

$$f_p(x) = (x+1)^p - (x-1)^p. \quad (25)$$

This formula has been originally derived in [31]. Note that $\text{Tr}\rho_\gamma^p$ is independent of the displacement m . We can verify that $\ln f_p$ is increasing and concave (Appendix A), so that $F_p(\nu) = \prod_{j=1}^n f_p(\nu_j)$ is increasing and Schur-concave (Appendix B). In Eq. (23), \mathcal{G} can be replaced by \mathcal{G}_p , the set of all pure Gaussian states [17]. In terms of F_p , we have

$$\left(\frac{2^n}{\xi_p(\Phi)} \right)^p = \inf_{\gamma_p} F_p \left[\nu(\phi(\gamma_p)) \right]. \quad (26)$$

The Gaussian minimal output entropy is defined as

$$S_{\min}(\Phi) = \inf_{\rho \in \mathcal{G}} S(\Phi(\rho)), \quad (27)$$

where $S(\rho) = -\text{Tr}\rho \ln \rho$ is the von Neumann entropy. Followed by the arguments presented in [17], it can be shown that \mathcal{G} in Eq. (27) can be replaced by \mathcal{G}_p . Since

$$\lim_{p \rightarrow 1+} \frac{d}{dp} \|\rho\|_p = -S(\rho), \quad (28)$$

$S_{\min}(\Phi)$ can be computed through $\xi_p(\Phi)$. Note that $S_{\min}(\Phi)$ is also independent of the displacement m . Hereafter, we have occasions to write the von Neumann entropy as $S(\gamma)$ instead of $S(\rho_\gamma)$ when we are dealing with Gaussian states.

By definition, the Holevo capacity for Gaussian state inputs or the Gaussian Holevo capacity is written as [32]

$$C_G(\Phi) = \sup_{\mu, \rho_{(\gamma, m)}} \left[S(\Phi(\bar{\rho})) - \int \mu(d\gamma, dm) S(\Phi(\rho_{(\gamma, m)})) \right], \quad (29)$$

where

$$\bar{\rho} = \int \mu(d\gamma, dm) \rho_{(\gamma, m)} \quad (30)$$

is the averaged signal state. In Eq. (29), the supremum is taken over all possible probability measures μ and signal states $\rho_{(\gamma, m)}$ constituting the signal ensemble. Since the states $\rho_{(\gamma, m)}$ are infinite dimensional states, the right-hand side of Eq. (29) becomes any large number if we do not impose some constraint on the signal states. Here, we take the energy constraint,

$$\text{Tr}\bar{\rho}H = \mathcal{E} \quad (31)$$

with

$$H = \sum_{k=1}^n \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right) = \frac{1}{2} \sum_{k=1}^n \omega_k (R_{2k-1}^2 + R_{2k}^2). \quad (32)$$

Here we recall that the von Neumann entropy of a Gaussian state depends only on the covariance matrix, and that channel Φ affects only the covariance matrix. Therefore, if we find a single state $\rho_{(\gamma^*, m)}$ that minimizes $S(\Phi(\rho))$, all possible Gaussian states ρ with the covariance matrix, γ^* , also minimizes $S(\Phi(\rho))$. This observation indicates that the optimal signal ensemble that attains the Gaussian Holevo capacity consists of Gaussian states with the common covariance matrix γ^* and a certain probability distribution of the displacement m . If we restrict the signal ensemble to that described above, it suffices to take a Gaussian probability distribution for the probability measure $\mu(d\gamma, dm) = \mu(dm)$. This is shown as follows [33]. If $\mu(dm)$ is a Gaussian distribution;

$$\mu(dm) = \frac{1}{\pi^n \sqrt{\det Y_\mu}} \exp(-m^T Y_\mu^{-1} m) dm \quad (33)$$

with $Y_\mu > 0$, the averaged input signal state is calculated as

$$\bar{\rho} = \int \mu(dm) \rho_{(\gamma, m)} = \rho_{(\gamma + Y_\mu, 0)}. \quad (34)$$

That is, $\bar{\rho}$ is also a Gaussian state with the covariance matrix $\bar{\gamma} = \gamma + Y_\mu$ and has the vanishing displacement. Equation (34) even holds for $Y_\mu \geq 0$. Since the displacement of $\bar{\rho}$ is zero,

$$\text{Tr} \bar{\rho} H = \frac{1}{2} \sum_{k=1}^n \omega_k (\langle R_{2k-1}^2 \rangle + \langle R_{2k}^2 \rangle) = \frac{1}{4} \sum_{k=1}^n \omega_k \text{Tr} \bar{\gamma}_{[k]}, \quad (35)$$

where $\bar{\gamma}_{[k]}$ denotes the principal submatrix of the k th mode of $\bar{\gamma}$ defined by Eq. (8) so that the energy constraint [(31)] is written as

$$\sum_{k=1}^n \omega_k \text{Tr}(\gamma + Y_\mu)_{[k]} = 4\mathcal{E}. \quad (36)$$

Since such a signal ensemble described above is not always optimal, we have

$$C_G(\Phi, \mathcal{E}) \geq \sup_{\gamma, Y_\mu(\geq 0)} [S(\Phi(\gamma + Y_\mu)) - S(\Phi(\gamma))], \quad (37)$$

where the supremum is taken under the constraint (36) so that the right-hand side of Eq. (37) is written as

$$\sup_{\gamma} S(\Phi[\gamma]) - \inf_{\gamma} S(\Phi[\gamma]) = \sup_{\gamma} S(\Phi[\gamma]) - S_{\min}(\Phi), \quad (38)$$

where the supremum is taken under the constraint

$$\sum_{k=1}^n \text{Tr} \omega_k \gamma_{[k]} = 4\mathcal{E}. \quad (39)$$

Here, we note the following extremal property of Gaussian states. For a given covariance matrix γ , the von Neumann entropy is maximized for the Gaussian state [33, 34]. Therefore,

for the Gaussian signal state $\rho_{(\gamma, m)}$ and the probability measure $\mu(d\gamma, dm)$, the quantity within the brackets of the right-hand side of Eq. (29) cannot exceed the value of the right-hand side of (37). Therefore, the equality holds in the inequality (37);

$$C_G(\Phi, \mathcal{E}) = \sup_{\gamma} S(\Phi(\gamma)) - S_{\min}(\Phi). \quad (40)$$

Again, the supremum is taken under the constraint [Eq. (39)].

In the above arguments, we have assumed implicitly that there exists a Gaussian state with the covariance matrix satisfying Eq. (39). Otherwise, $C_G(\Phi, \mathcal{E})$ should be zero because \mathcal{E} in the energy constraint (31) would be smaller than the sum of the energies of the zero-point oscillations over the modes.

IV. ADDITIVITY AND MULTIPLICATIVITY PROBLEMS OF GAUSSIAN CHANNELS

For the tensor product of the Gaussian channels, $\Phi = \bigotimes_{j=1}^m \Phi_j$, it is evident from the definition that

$$\xi_p(\Phi) \geq \prod_{j=1}^m \xi_p(\Phi_j). \quad (41)$$

If the equality holds in the inequality (41), we say that the maximal output p norm is multiplicative for Gaussian channels Φ_j . To show the multiplicativity of the maximal output p norm, it suffices to show

$$\inf_{\gamma_p} F_p[\nu(\Phi(\gamma_p))] = \prod_{j=1}^m \inf_{\gamma_p} F_p[\nu(\Phi_j(\gamma_p))]. \quad (42)$$

In Eq. (42), γ_p in the left-hand side of the equation is the covariance matrix of a pure Gaussian state on the composite Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$, while γ_p in the right-hand side of the equation is the covariance matrix of a pure Gaussian state on the Hilbert space \mathcal{H}_j .

By noting Eq. (28), it follows that if the maximal output p norm is multiplicative, then the minimal output entropy is additive.

Let \mathcal{E} be the value for the energy constraint [Eq. (31)] for the Gaussian channel Φ_j and $\mathcal{E} = \sum_{j=1}^m \mathcal{E}_j$. From the definition, the Gaussian Holevo capacity of the tensor product channel is greater than or equal to the supremum of the sum of the Gaussian Holevo capacity of individual channels;

$$C_G(\Phi, \mathcal{E}) \geq \sup_{\{\mathcal{E}_j\}, \sum_{j=1}^m \mathcal{E}_j = \mathcal{E}} \sum_{j=1}^m C_G(\Phi_j, \mathcal{E}_j). \quad (43)$$

Here, the supremum is taken over all possible combinations of \mathcal{E}_j under the constraint $\sum_{j=1}^m \mathcal{E}_j = \mathcal{E}$. If the equality holds in the inequality (43), we say that the energy-constrained Gaussian Holevo capacity is additive for Gaussian channels Φ_j . Now let ρ be a Gaussian state on the composite Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ and define $\rho_j = \text{Tr}_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{j-1} \otimes \mathcal{H}_{j+1} \otimes \cdots \otimes \mathcal{H}_m} \rho$. By noting the subadditivity of the von Neumann entropy [35],

$S(\rho) \leq \sum_{j=1}^m S(\rho_j)$, we have

$$S(\phi(\gamma)) \leq \sum_{j=1}^m S(\phi_j(\gamma_j)), \quad (44)$$

where γ_j denotes the covariance matrix of the Gaussian state ρ_j . Therefore, if the minimal output entropy $S_{\min}(\Phi)$ is additive for the channels Φ_j , then

$$C_G(\Phi, \mathcal{E}) \leq \sup_{\{\mathcal{E}_j\}, \sum_{j=1}^m \mathcal{E}_j = \mathcal{E}} \sum_{j=1}^m C_G(\Phi_j, \mathcal{E}_j). \quad (45)$$

This implies the additivity of the energy-constrained Gaussian Holevo capacity

$$C_G(\Phi, \mathcal{E}) = \sup_{\{\mathcal{E}_j\}, \sum_{j=1}^m \mathcal{E}_j = \mathcal{E}} \sum_{j=1}^m C_G(\Phi_j, \mathcal{E}_j). \quad (46)$$

Serafini *et al.* [17] proved that the Gaussian maximal output p norm of a tensor product of identical single mode Gaussian channels and that of single mode channels described by X_i and Y_i [(21)] such that $\det X_i$ are identical and $Y_i > 0$ for all i , are multiplicative under Gaussian state inputs for $p > 1$. Consequently, the Gaussian minimal output entropy and energy-constrained Gaussian Holevo capacity are additive for such tensor product channels.

V. A MAJORIZATION RELATION ON SYMPLECTIC EIGENVALUES

Lemma 1. – Let A be a $2n \times 2n$ real symmetric positive-definite matrix ($A = A^T > 0$). Then

$$\min_{S J_n S^T = J_k} \text{Tr} S A S^T = 2 \sum_{j=1}^k v_j^\uparrow(A), \quad 1 \leq k \leq n. \quad (47)$$

The minimum in Eq. (47) is taken over all $2k \times 2n$ real matrices S satisfying $S J_n S^T = J_k$.

Proof. First of all, we note that a $2k \times 2n$ matrix S satisfying $S J_n S^T = J_k$ is the first $2k$ rows of a symplectic transformation, $S_n \in \text{Sp}(2n, \mathbb{R})$, and that A can be written in the Williamson standard form to obtain

$$\min_{S J_n S^T = J_k} \text{Tr} S A S^T = \min_{S_n J_n S_n^T = J_n} \sum_{j=1}^{2k} (S_n D_A S_n^T)_{j,j}, \quad (48)$$

where we have used the fact that a product of symplectic transformations is a symplectic transformation and have defined $D_A = \text{diag}[v_1^\uparrow(A), v_1^\uparrow(A), \dots, v_n^\uparrow(A), v_n^\uparrow(A)]$. Here, we write S_n

in the Euler decomposition form [Eq. (13)] to obtain

$$\begin{aligned} \sum_{j=1}^{2k} (S_n D_A S_n^T)_{j,j} &= \sum_{j=1}^k \sum_{l,m=1}^n P_{2l-1,2m-1}^{(j)} z_l z_m Q_{2l-1,2m-1} \\ &\quad + \sum_{j=1}^k \sum_{l,m=1}^n P_{2l-1,2m}^{(j)} z_l z_m^{-1} Q_{2l-1,2m} \\ &\quad + \sum_{j=1}^k \sum_{l,m=1}^n P_{2l,2m-1}^{(j)} z_l^{-1} z_m Q_{2l,2m-1} \\ &\quad + \sum_{j=1}^k \sum_{l,m=1}^n P_{2l,2m}^{(j)} z_l^{-1} z_m^{-1} Q_{2l,2m}, \end{aligned} \quad (49)$$

where $z_j \geq 1$ ($j = 1, 2, \dots, n$),

$$P_{l,m}^{(j)} = T_{2j-1,l}^{(1)} T_{2j-1,m}^{(1)} + T_{2j,l}^{(1)} T_{2j,m}^{(1)}, \quad (50)$$

and

$$Q_{l,m} = \sum_{p=1}^n v_p^\uparrow(A) (T_{l,2p-1}^{(2)} T_{m,2p-1}^{(2)} + T_{l,2p}^{(2)} T_{m,2p}^{(2)}) \quad (51)$$

with $T^{(1)}, T^{(2)} \in K(n)$. Using Eqs. (15) and (16) for $T^{(1)}$ and $T^{(2)}$, the elements of $P^{(j)}$ and Q are computed through the elements of $n \times n$ unitary matrices U and V as follows:

$$P_{2l-1,2m-1}^{(j)} = P_{2l,2m}^{(j)} = u_{j,l}^R u_{j,m}^R + u_{j,l}^I u_{j,m}^I, \quad (52)$$

$$P_{2l-1,2m}^{(j)} = -P_{2l,2m-1}^{(j)} = u_{j,l}^R u_{j,m}^I - u_{j,l}^I u_{j,m}^R, \quad (53)$$

$$Q_{2l-1,2m-1} = Q_{2l,2m} = \sum_{p=1}^n v_p^\uparrow(A) (v_{l,p}^R v_{m,p}^R + v_{l,p}^I v_{m,p}^I), \quad (54)$$

and

$$Q_{2l-1,2m} = -Q_{2l,2m-1} = \sum_{p=1}^n v_p^\uparrow(A) (-v_{l,p}^R v_{m,p}^I + v_{l,p}^I v_{m,p}^R). \quad (55)$$

Substituting Eqs. (52), (53), (54), and (55) into Eq. (49) yields

$$\begin{aligned} &\sum_{j=1}^{2k} (S_n D_A S_n^T)_{j,j} \\ &= \frac{1}{4} \sum_{j=1}^k \sum_{l,m=1}^n u_{j,l} u_{j,m}^* (z_l + z_l^{-1})(z_m + z_m^{-1}) \sum_{p=1}^n v_p^\uparrow(A) v_{l,p} v_{m,p}^* \\ &\quad + \frac{1}{4} \sum_{j=1}^k \sum_{l,m=1}^n u_{j,l}^* u_{j,m} (z_l + z_l^{-1})(z_m + z_m^{-1}) \sum_{p=1}^n v_p^\uparrow(A) v_{l,p}^* v_{m,p} \\ &\quad + \frac{1}{4} \sum_{j=1}^k \sum_{l,m=1}^n u_{j,l} u_{j,m}^* (z_l - z_l^{-1})(z_m - z_m^{-1}) \sum_{p=1}^n v_p^\uparrow(A) v_{l,p}^* v_{m,p} \\ &\quad + \frac{1}{4} \sum_{j=1}^k \sum_{l,m=1}^n u_{j,l}^* u_{j,m} (z_l - z_l^{-1})(z_m - z_m^{-1}) \sum_{p=1}^n v_p^\uparrow(A) v_{l,p} v_{m,p}^* \\ &= \sum_{l=1}^4 \sum_{j=1}^k (W^{(l)} D_A W^{(l)\dagger})_{j,j}, \end{aligned} \quad (56)$$

where $*$ means complex conjugate. In the right-hand side of Eq. (56), $W^{(1)} = UZ_+V$, $W^{(2)} = U^*Z_+V^*$, $W^{(3)} = UZ_-V^*$, and $W^{(4)} = U^*Z_-V$, with

$$Z_{\pm} = \frac{1}{2} \text{diag}(z_1 \pm z_1^{-1}, \dots, z_n \pm z_n^{-1}). \quad (57)$$

Note that matrices $W^{(l)}D_A W^{(l)\dagger}$ are positive semidefinite; $W^{(l)}D_A W^{(l)\dagger} \geq 0$ ($l = 1, \dots, 4$). Now let $\lambda_j^\uparrow(W^{(l)}D_A W^{(l)\dagger})$ be eigenvalues of Hermitian matrices $W^{(l)}D_A W^{(l)\dagger}$. By the Schur theorem (Appendix B), we have

$$\sum_{j=1}^k (W^{(l)}D_A W^{(l)\dagger})_{j,j}^\uparrow \geq \sum_{j=1}^k \lambda_j^\uparrow(W^{(l)}D_A W^{(l)\dagger}), \quad l = 1, \dots, 4. \quad (58)$$

Thus, we obtain

$$\begin{aligned} \sum_{j=1}^{2k} (S_n D_A S_n^T)_{j,j} &\geq \sum_{l=1}^4 \sum_{j=1}^k (W^{(l)}D_A W^{(l)\dagger})_{j,j}^\uparrow \\ &\geq \sum_{l=1}^4 \sum_{j=1}^k \lambda_j^\uparrow(W^{(l)}D_A W^{(l)\dagger}). \end{aligned} \quad (59)$$

Here,

$$\lambda_j^\uparrow(W^{(1)}D_A W^{(1)\dagger}) = \lambda_j^\uparrow(Z_+ V D_A V^\dagger Z_+) = \lambda_j^\uparrow(Z_+ C Z_+) \quad (60)$$

with $C = V D_A V^\dagger \geq 0$. The first equality is due to the unitary invariance of the eigenvalues of the Hermitian matrices. The eigenvalues, $\lambda_j^\uparrow(Z_+ C Z_+)$, admit the following max-min representation [36]:

$$\lambda_j^\uparrow(Z_+ C Z_+) = \max_{w_1, \dots, w_{j-1} \in \mathbb{C}^n} \min_{\substack{x \neq 0 \\ x \perp w_1, \dots, w_{j-1}}} \frac{x^\dagger Z_+ C Z_+ x}{\|x\|_2^2}. \quad (61)$$

If we write $y = Z_+ x$,

$$\|x\|_2^2 = \sum_{j=1}^n (Z_+^{-1})_j^2 |y_j|^2 \leq \sum_{j=1}^n |y_j|^2 = \|y\|_2^2. \quad (62)$$

Hence,

$$\begin{aligned} \lambda_j^\uparrow(Z_+ C Z_+) &\geq \max_{w_1, \dots, w_{j-1} \in \mathbb{C}^n} \min_{\substack{y \neq 0 \\ y \perp Z_+^{-1} w_1, \dots, Z_+^{-1} w_{j-1}}} \frac{y^\dagger C y}{\|y\|_2^2} \\ &= \lambda_j^\uparrow(C) = \nu_j^\uparrow(A) \end{aligned} \quad (63)$$

so that $\lambda_j^\uparrow(W^{(1)}D_A W^{(1)\dagger}) \geq \nu_j^\uparrow(A)$. Similarly, $\lambda_j^\uparrow(W^{(2)}D_A W^{(2)\dagger}) \geq \nu_j^\uparrow(A)$. Since $\lambda_j^\uparrow(W^{(3)}D_A W^{(3)\dagger}) \geq 0$ and $\lambda_j^\uparrow(W^{(4)}D_A W^{(4)\dagger}) \geq 0$, we find

$$\sum_{j=1}^{2k} (S_n D_A S_n^T)_{j,j} \geq 2 \sum_{j=1}^k \nu_j^\uparrow(A). \quad (64)$$

The equality holds for $S_n = \mathbf{I}_{2n} \in \text{Sp}(2n, \mathbb{R})$. This completes the proof. \square

Theorem 1. – Let A and B be $2n \times 2n$ real positive symmetric matrices ($A = A^T > 0$, $B = B^T > 0$). Then

$$\nu(A+B) <^w \nu(A) + \nu(B). \quad (65)$$

Proof. By Lemma 1, we have

$$\begin{aligned} 2 \sum_{j=1}^k \nu_j^\uparrow(A+B) &= \min_{S J_n S^T = J_k} \text{Tr} S(A+B) S^T \\ &\geq \min_{S J_n S^T = J_k} \text{Tr} S A S^T + \min_{S J_n S^T = J_k} \text{Tr} S B S^T \\ &= 2 \sum_{j=1}^k \nu_j^\uparrow(A) + 2 \sum_{j=1}^k \nu_j^\uparrow(B) \\ &= 2 \sum_{j=1}^k [\nu(A) + \nu(B)]_j^\uparrow. \end{aligned} \quad (66)$$

By definition of the weak supermajorization [(B2)], inequality (66) yields the desired relation (65). \square

VI. ADDITIVITY AND MULTIPLICATIVITY PROPERTIES OF GAUSSIAN CHANNELS

In this section, we focus on two classes of Gaussian channels; the classical noise channel and the thermal noise channel. Both are important cases of Gaussian channels.

In the classical noise channel, a classical Gaussian noise is added to the input states. Since $\mathcal{W}(\xi) R_j \mathcal{W}^\dagger(\xi) = R_j + \xi_j$, the classical noise channel is described by

$$\begin{aligned} \Phi^{cl}(\rho_{(\gamma, m)}) &= \frac{1}{\pi^n \sqrt{\det Y}} \int d^{2n} \xi \exp(-\xi^T Y^{-1} \xi) \mathcal{W}(\xi) \rho_{(\gamma, m)} \mathcal{W}^\dagger(\xi) \\ &= \rho_{(\gamma+Y, m)} \end{aligned} \quad (67)$$

with $Y \geq 0$. Namely, the transformations of covariance matrix is given by

$$\phi^{cl}(\gamma) = \gamma + Y. \quad (68)$$

In the thermal noise channel, the signal Gaussian states interact with an environment that is in thermal equilibrium. This channel is modeled by beamsplitters that couple the input Gaussian state and the thermal reservoir. Let a_j and b_j be annihilation operators of the j th mode of the signal state ρ and the thermal state ρ_{th} that acts as a thermal reservoir. The action of the beamsplitter is described by the transformations, $a_j \mapsto \cos \theta_j a_j + \sin \theta_j b_j$ and $b_j \mapsto -\sin \theta_j a_j + \cos \theta_j b_j$. Accordingly, the corresponding symplectic transformation takes the form,

$$S_j = \begin{pmatrix} \cos \theta_j \mathbf{I}_2 & \sin \theta_j \mathbf{I}_2 \\ -\sin \theta_j \mathbf{I}_2 & \cos \theta_j \mathbf{I}_2 \end{pmatrix}. \quad (69)$$

Therefore, the output Gaussian state has the covariance matrix,

$$\phi^{th}(\gamma) = \text{Tr}_{th}[S^{-1}(\gamma \oplus \gamma_{th})(S^{-1})^T], \quad (70)$$

where $S = S_1 \oplus S_2 \oplus \dots \oplus S_n$, $\gamma_{th} = \bigoplus_{j=1}^n (2\langle n \rangle_j + 1)\mathbf{I}_2$ denotes the covariance matrix of the thermal state ρ_{th} with $\langle n \rangle_j$ being the averaged photon number of the j th mode, and Tr_{th} describes the trace over the thermal state. Using Eq. (69), the right-hand side of Eq. (70) is calculated as

$$\phi^{th}(\gamma) = X^T \gamma X + Y \quad (71)$$

where

$$X = \bigoplus_{j=1}^n \sqrt{\eta_j} \mathbf{I}_2 \quad (72)$$

and

$$Y = \bigoplus_{j=1}^n (2\langle n \rangle_j + 1)(1 - \eta_j) \mathbf{I}_2 \quad (73)$$

with $\eta_j = \cos^2 \theta_j$ being the transmittivity of the beamsplitter. At zero temperature ($\langle n \rangle_j = 0$), the thermal noise channel is reduced to the lossy or attenuation channel [25, 26].

A. Classical noise channels

For the l_j -mode classical noise channel Φ_j^{cl} , the covariance matrix γ is transformed according to $\phi_j^{cl}(\gamma) = \gamma + Y_j$, with $Y_j \geq 0$. The tensor product of Φ_j^{cl} is also a classical noise channel, and the covariance matrix of the output is given by

$$\phi^{cl}(\gamma) = \bigoplus_{j=1}^m \phi_j^{cl}(\gamma) = \gamma + Y, \quad (74)$$

where γ is the covariance matrix of the input Gaussian state and $Y = \bigoplus_{j=1}^m Y_j$. Since Y_j is not always strictly positive definite, we add $\varepsilon \mathbf{I}_{2l_j}$ to Y_j ($\varepsilon > 0$); $Y_j(\varepsilon) = Y_j + \varepsilon \mathbf{I}_{2l_j} > 0$ ($j = 1, \dots, m$) so that we can apply the Williamson theorem to $Y_j(\varepsilon)$. Here, we write $Y(\varepsilon) = \bigoplus_{j=1}^m Y_j(\varepsilon)$. By Williamson theorem, there exists $S_j \in \text{Sp}(2l_j, \mathbb{R})$ such that

$$\begin{aligned} S_j Y(\varepsilon)_j S_j^T &= \text{diag}(y_1^{(j)}(\varepsilon), y_1^{(j)}(\varepsilon), \dots, y_{l_j}^{(j)}(\varepsilon), y_{l_j}^{(j)}(\varepsilon)) \\ &= D_{Y_j}(\varepsilon). \end{aligned} \quad (75)$$

Here, we write $S = \bigoplus_{j=1}^m S_j$ so that $S Y(\varepsilon) S^T = \bigoplus_{j=1}^m D_{Y_j}(\varepsilon) = D_Y(\varepsilon)$. By Theorem 1, we have

$$\nu[\gamma + Y(\varepsilon)] = \nu[S \gamma S^T + D_Y(\varepsilon)] <^w \nu(\gamma) + \nu(D_Y(\varepsilon)). \quad (76)$$

Since F_p is increasing and Schur-concave, (76) yields

$$F_p[\nu(\gamma + Y(\varepsilon))] \geq F_p(\nu(\gamma) + \nu(D_Y(\varepsilon))). \quad (77)$$

Here we can take the limit $\varepsilon \rightarrow 0$ to obtain

$$\inf_{\gamma_p} F_p[\nu(\phi^{cl}(\gamma))] \geq \inf_{\gamma_p} F_p(\nu(\gamma_p) + \nu(Y)), \quad (78)$$

where $\nu(Y) = \bigoplus_{j=1}^m \nu(Y_j)$ with

$$\begin{aligned} \nu(Y_j) &= \lim_{\varepsilon \rightarrow 0} (y_1^{(j)}(\varepsilon), y_2^{(j)}(\varepsilon), \dots, y_{l_j}^{(j)}(\varepsilon)) \\ &= (y_1^{(j)}, y_2^{(j)}, \dots, y_{l_j}^{(j)}). \end{aligned} \quad (79)$$

The infimum in the right-hand side of (78) is achieved for $\nu(\gamma_p) = (1, 1, \dots, 1, 1)$ and the equality holds if $S \gamma S^T$ takes the Williamson standard form. Namely, for the covariance matrix γ_p such that $S \gamma_p S^T = \text{diag}(1, 1, \dots, 1, 1)$,

$$\begin{aligned} \inf_{\gamma_p} F_p[\nu(\phi^{cl}(\gamma_p))] &= \prod_{j=1}^m \prod_{k=1}^{l_j} f_p(1 + y_k^{(j)}) \\ &= \prod_{j=1}^m \inf_{\gamma_p} F_p[\nu(\phi_j^{cl}(\gamma_p))]. \end{aligned} \quad (80)$$

That is, the maximal output p norm is multiplicative. Consequently, the Gaussian minimal output entropy and the Gaussian Holevo capacity are additive. Note that S defined above is the direct sum of local symplectic transformations and $\text{diag}(1, 1, \dots, 1, 1)$ is the covariance matrix of the pure separable state so that the optimal $\gamma_p = S^{-1} \text{diag}(1, 1, \dots, 1, 1) (S^T)^{-1}$ is a separable pure state. This observation also indicates the multiplicativity of the maximal output p norm.

B. Thermal noise channels

For the l_j -mode thermal noise channel Φ_j^{th} , the covariance matrix is transformed according to

$$\phi_j^{th}(\gamma) = \phi_j^{(0)}(\gamma) + Y_j, \quad (81)$$

where

$$Y_j = \bigoplus_{k=1}^{l_j} (2\langle n \rangle_k + 1)(1 - \eta_k) \mathbf{I}_2 \quad (82)$$

and

$$\phi_j^{(0)}(\gamma) = X^T \gamma X + Y, \quad (83)$$

with $X = \bigoplus_{k=1}^{l_j} \sqrt{\eta_k} \mathbf{I}_2$, $Y = \bigoplus_{k=1}^{l_j} (1 - \eta_k) \mathbf{I}_2$ and $0 \leq \eta_j \leq 1$. The tensor product of Φ_j^{th} is a Gaussian channel and the covariance matrix of the output state is given by

$$\phi^{th}(\gamma) = \bigoplus_{j=1}^m \phi_j^{th}(\gamma) = \phi^{(0)}(\gamma) + Y, \quad (84)$$

where $\phi^{(0)}(\gamma) = \bigoplus_{j=1}^m \phi_j^{(0)}(\gamma)$ and $Y = \bigoplus_{j=1}^m Y_j$. Again, we add $\varepsilon \mathbf{I}_{2n}$ to Y ($\varepsilon > 0$); $Y(\varepsilon) = Y + \varepsilon \mathbf{I}_{2n}$ to ensure $Y(\varepsilon) > 0$ ($n = \sum_{j=1}^m l_j$). Accordingly, we write $\phi^{th}(\gamma, \varepsilon) = \phi^{(0)}(\gamma) + Y(\varepsilon)$. By Theorem 1, we have

$$\nu[\phi^{th}(\gamma, \varepsilon)] <^w \nu(\phi^{(0)}(\gamma)) + \nu(Y(\varepsilon)). \quad (85)$$

Since F_p is increasing and Schur-concave, (85) yields

$$F_p\{\nu[\phi^{th}(\gamma, \varepsilon)]\} \geq F_p[\nu(\phi^{(0)}(\gamma)) + \nu(Y(\varepsilon))], \quad (86)$$

Here we can take the limit $\varepsilon \rightarrow 0$ to obtain

$$\inf_{\gamma_p} F_p[\nu(\phi^{th}(\gamma_p))] \geq \inf_{\gamma_p} F_p[\nu(\phi^{(0)}(\gamma_p)) + \nu(Y)]. \quad (87)$$

Since the channel $\phi^{(0)}(\gamma)$ is completely positive, the Gaussian state with the covariance matrix $\phi^{(0)}(\gamma_p)$ is a physical state so that $\nu_j(\phi^{(0)}(\gamma_p)) \geq 1$ ($j = 1, 2, \dots, n$). For $\gamma = \text{diag}(1, 1, \dots, 1, 1)$, $\nu_j(\phi^{(0)}(\gamma)) = 1$ and the equality holds in (87). Hence,

$$\begin{aligned} \inf_{\gamma_p} F_p[\nu(\phi^{th}(\gamma_p))] &= \prod_{j=1}^m \prod_{k=1}^{l_j} f_p[1 + 2(1 - \eta_k) \langle n \rangle_k] \\ &= \prod_{j=1}^m \inf_{\gamma_p} F_p[\nu(\phi_j^{th}(\gamma_p))]. \end{aligned} \quad (88)$$

That is, the maximal output p norm is multiplicative. Consequently, the Gaussian minimal output entropy and the energy-constrained Gaussian Holevo capacity are additive.

VII. CONCLUDING REMARKS

We proved the multiplicativity of maximal output p norm of classical noise channels and that of thermal noise channels of arbitrary modes for all $p > 1$ under the assumption that the input signal states were Gaussian states. As a direct consequence, we also proved the additivity of the minimal output entropy and the energy-constrained Holevo capacity for those Gaussian channels under Gaussian inputs. A majorization relation on symplectic eigenvalues was of importance in the proof.

At present, very little is known about the inequalities related to symplectic eigenvalues of real positive-definite matrices. Efforts to unveil such unknown relations would assist in the analysis of entropic quantities of Gaussian states and would also shed light on the properties of Gaussian state entanglement [37] and secure communication via Gaussian channels [38].

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APPENDIX A: CONCAVITY OF $\ln f_p(x)$

It is readily seen that $f_p(x)$ is concave for $1 \leq p \leq 2$ and is convex for $p \geq 2$. Therefore, $\ln f_p(x)$ is concave for $1 \leq p \leq 2$

2. In order to show the concavity of $\ln f_p(x)$ for $p \geq 2$, we examine the second derivative of $\ln f_p(x)$;

$$\frac{d^2}{dx^2} \ln f_p(x) = -\frac{p}{f_p^2(x)} g_p(x), \quad (A1)$$

where $g_p(x) = 4p(x^2 - 1)^{p-2} + f_p(x)f_{p-2}(x)$. For $p \geq 2$, we find that $g_p(x) \geq 0$ so that $d^2 \ln f_p(x)/dx^2 \leq 0$. That is, $\ln f_p(x)$ is concave for $p \geq 2$. Thus, $\ln f_p(x)$ is concave for all $p \geq 1$.

APPENDIX B: MAJORIZATION AND SCHUR CONVEXITY

In this Appendix, we present definitions and basic facts on majorization and the Schur convexity (concavity) used in this paper [39].

For vectors, $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ ($x_j, y_j \in \mathbb{R}$), we write $x \leq y$ if $x_j \leq y_j$ ($j = 1, 2, \dots, n$). Let $x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \dots, x_n^\downarrow)$ denote the decreasing rearrangement of x , where $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_n^\downarrow$. Similarly, let $x^\uparrow = (x_1^\uparrow, x_2^\uparrow, \dots, x_n^\uparrow)$ denote the increasing rearrangement of x , where $x_1^\uparrow \leq x_2^\uparrow \leq \dots \leq x_n^\uparrow$.

We say that x is majorized by y and write $x < y$ if

$$\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad k = 1, 2, \dots, n \quad (B1)$$

with the equality for $k = n$.

We say that x is weakly submajorized (weakly supermajorized) by y and write $x <_w$ ($<^w$) y if

$$\sum_{j=1}^k x_j^{\downarrow(\uparrow)} \leq (\geq) \sum_{j=1}^k y_j^{\downarrow(\uparrow)}, \quad k = 1, 2, \dots, n. \quad (B2)$$

It is easy to see that $x < y$ if and only if $x <_w y$ and $x <^w y$.

A real-valued function f defined on \mathbb{R}^n is said to be increasing if $x \leq y \Rightarrow f(x) \leq f(y)$, while f is said to be decreasing if $-f$ is increasing.

A real-valued function f defined on \mathbb{R}^n is said to be Schur-convex if $x < y \Rightarrow f(x) \leq f(y)$ while f is said to be Schur-concave if $-f$ is Schur-convex.

A real-valued function f defined on \mathbb{R}^n satisfied $x <_w$ ($<^w$) $y \Rightarrow f(x) \leq (\geq) f(y)$ if and only if f is increasing and Schur-convex (concave).

Let g be a continuous and nonnegative function on \mathbb{R} . Then, $f(x) = \prod_{j=1}^n g(x_j)$ is Schur-convex (concave) if and only if $\log g$ is convex (concave).

An application of majorization theory to matrix analysis is the following Schur theorem [40]. Let $A \in M_n(\mathbb{C})$ be an Hermitian matrix. Let $\text{diag}(A)$ denote the vector whose elements are the diagonal entries of A and $\lambda(A)$ the vector whose coordinates are eigenvalues of A . Then, $\text{diag}(A) < \lambda(A)$.

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